Online Appendix for

"A Global Equilibrium Asset Pricing Model with Home Preference"

Bruno Solnik and Luo Zuo

June 16, 2011

This online appendix contains various derivations for the published article. It is organized as follows. Section A proves the concavity of the expected utility function. Section B derives the optimal global asset allocation with home preference. Section C examines the global market equilibrium and derives the asset pricing relation.

A. *Concavity of EU_i with respect to* $\boldsymbol{\alpha}_\mathbf{i}$

Recall that:

$$U_i(W_i^1) = v_i(W_i\boldsymbol{\alpha}_\mathbf{i}^T\mathbf{r}) + f_i(v_i(W_i\boldsymbol{\alpha}_\mathbf{i}^T\mathbf{r}) - v_i(W_i\mathbf{d}_\mathbf{i}^T\mathbf{r})) \quad \text{(A1)}$$

$$EU_i(W_i^1) = Ev_i(W_i\boldsymbol{\alpha}_\mathbf{i}^T\mathbf{r}) + Ef_i(v_i(W_i\boldsymbol{\alpha}_\mathbf{i}^T\mathbf{r}) - v_i(W_i\mathbf{d}_\mathbf{i}^T\mathbf{r})) \quad \text{(A2)}$$

We first show that U_i is concave in $\boldsymbol{\alpha}_\mathbf{i}$. Let's take derivatives of U_i with respect to $\boldsymbol{\alpha}_\mathbf{i}$.

The gradient is:

$$\nabla U_i = \frac{\partial U_i}{\partial \boldsymbol{\alpha}_\mathbf{i}} = W_i\mathbf{r}v_i' + W_i\mathbf{r}f_i' \times v_i' = W_i\mathbf{r}v_i'(1+f_i')$$

where the argument of v_i' is $W_i\boldsymbol{\alpha}_\mathbf{i}^T\mathbf{r}$, and the argument of f_i' is $v_i(W_i\boldsymbol{\alpha}_\mathbf{i}^T\mathbf{r}) - v_i(W_i\mathbf{d}_\mathbf{i}^T\mathbf{r})$. So $\frac{\partial U_i}{\partial \boldsymbol{\alpha}_\mathbf{i}}$ is continuous.

The Hessian matrix is:

$$D^2U_i = \frac{\partial^2 U_i}{\partial \boldsymbol{\alpha}_\mathbf{i} \partial \boldsymbol{\alpha}_\mathbf{i}^T} = W_i^2\mathbf{r}\mathbf{r}^T\left[v_i''(1+f_i') + v_i'^2 f_i''\right]$$

where v_i' and v_i'' are valued at $W_i\boldsymbol{\alpha}_\mathbf{i}^T\mathbf{r}$ and f_i' and f_i'' are valued at $v_i(W_i\boldsymbol{\alpha}_\mathbf{i}^T\mathbf{r}) - v_i(W_i\mathbf{d}_\mathbf{i}^T\mathbf{r})$.

The Hessian matrix D^2U_i is negative semi-definite for all values of $\boldsymbol{\alpha}_\mathbf{i}$ and $\mathbf{r}$, as $v_i', f_i' > 0$ and $v_i'', f_i'' < 0$. (Note that $\mathbf{r}\mathbf{r}^T$ is positive semi-definite.)

Let's now turn to EU_i which can be written as a function of $\boldsymbol{\alpha}_\mathbf{i}$. The second derivative of EU_i with respect to $\boldsymbol{\alpha}_\mathbf{i}$ is:

$$\frac{\partial^2 EU_i}{\partial\boldsymbol{\alpha}_\mathbf{i}\partial\boldsymbol{\alpha}_\mathbf{i}^T} = E\frac{\partial^2 U_i}{\partial\boldsymbol{\alpha}_\mathbf{i}\partial\boldsymbol{\alpha}_\mathbf{i}^T} = W_i^2 E(\mathbf{r}\mathbf{r}^T\left[v_i''(1+f_i') + v_i'^2\ f_i''\right])$$

Because $\frac{\partial^2 U_i}{\partial\boldsymbol{\alpha}_\mathbf{i}\partial\boldsymbol{\alpha}_\mathbf{i}^T}$ is negative semi-definite for all values of $\boldsymbol{\alpha}_\mathbf{i}$ and **r**, so is $\frac{\partial^2 EU_i}{\partial\boldsymbol{\alpha}_\mathbf{i}\partial\boldsymbol{\alpha}_\mathbf{i}^T}$.

B. Optimal global asset allocation with home preference

For a given allocation $\boldsymbol{\alpha}_\mathbf{i}$, we develop the Taylor expansion[1] around 0 for small price movements. We expand the value function $v_i(.)$ around 0 and the regret function $f_i(.)$ around 0. So the implicit arguments are 0 for all derivatives of $v_i(.)$ and $f_i(.)$. With the additional notations $\overline{\mathbf{r}} = E(\mathbf{r})$ and $\boldsymbol{\Omega} = E(\mathbf{r}\mathbf{r}^\mathbf{T})$, we get:

$$Ev_i(W_i\boldsymbol{\alpha}_\mathbf{i}^T\mathbf{r}) \approx v_i(0) + W_i\boldsymbol{\alpha}_\mathbf{i}^T\overline{\mathbf{r}}v_i' + \frac{W_i^2}{2}\left[\boldsymbol{\alpha}_\mathbf{i}^T\boldsymbol{\Omega}\boldsymbol{\alpha}_\mathbf{i}\right]v_i'' \qquad \text{(A3)}$$

and

$$\begin{aligned}&Ef_i(v_i(W_i\boldsymbol{\alpha}_\mathbf{i}^T\mathbf{r}) - v_i(W_i\mathbf{d}_\mathbf{i}^T\mathbf{r}))\\ &\approx E\left[v_i(W_i\boldsymbol{\alpha}_\mathbf{i}^T\mathbf{r}) - v_i(W_i\mathbf{d}_\mathbf{i}^T\mathbf{r})\right]f_i' + \frac{1}{2}E\left[v_i(W_i\boldsymbol{\alpha}_\mathbf{i}^T\mathbf{r}) - v_i(W_i\mathbf{d}_\mathbf{i}^T\mathbf{r})\right]^2 f_i''\end{aligned} \qquad \text{(A4)}$$

Note that:

$$v_i(W_i\boldsymbol{\alpha}_\mathbf{i}^T\mathbf{r}) = v_i(0) + W_i\boldsymbol{\alpha}_\mathbf{i}^T\mathbf{r}v_i' + \frac{W_i^2}{2}(\boldsymbol{\alpha}_\mathbf{i}^T\mathbf{r})^2 v_i''$$

$$v_i(W_i\mathbf{d}_\mathbf{i}^T\mathbf{r}) = v_i(0) + W_i\mathbf{d}_\mathbf{i}^T\mathbf{r}v_i' + \frac{W_i^2}{2}(\mathbf{d}_\mathbf{i}^T\mathbf{r})^2 v_i''$$

$$E\left[v_i(W_i\boldsymbol{\alpha}_\mathbf{i}^T\mathbf{r}) - v_i(W_i\mathbf{d}_\mathbf{i}^T\mathbf{r})\right] \approx W_i(\boldsymbol{\alpha}_\mathbf{i}^T - \mathbf{d}_\mathbf{i}^T)\overline{\mathbf{r}}v_i' + \frac{W_i^2}{2}\boldsymbol{\alpha}_\mathbf{i}^T\boldsymbol{\Omega}\boldsymbol{\alpha}_\mathbf{i}v_i'' - \frac{W_i^2}{2}\mathbf{d}_\mathbf{i}^T\boldsymbol{\Omega}\mathbf{d}_\mathbf{i}v_i''$$

$$E\left[v_i(W_i\boldsymbol{\alpha}_\mathbf{i}^T\mathbf{r}) - v_i(W_i\mathbf{d}_\mathbf{i}^T\mathbf{r})\right]^2 \approx W_i^2(\boldsymbol{\alpha}_\mathbf{i}^T - \mathbf{d}_\mathbf{i}^T)\boldsymbol{\Omega}(\boldsymbol{\alpha}_\mathbf{i} - \mathbf{d}_\mathbf{i})v_i'^2$$

Hence (A4) becomes:

$$\begin{aligned}&Ef_i(v_i(W_i\boldsymbol{\alpha}_\mathbf{i}^T\mathbf{r}) - v_i(W_i\mathbf{d}_\mathbf{i}^T\mathbf{r}))\\ &\approx \left[W_i(\boldsymbol{\alpha}_\mathbf{i}^T - \mathbf{d}_\mathbf{i}^T)\overline{\mathbf{r}}v_i' + \frac{W_i^2}{2}\boldsymbol{\alpha}_\mathbf{i}^T\boldsymbol{\Omega}\boldsymbol{\alpha}_\mathbf{i}v_i'' - \frac{W_i^2}{2}\mathbf{d}_\mathbf{i}^T\boldsymbol{\Omega}\mathbf{d}_\mathbf{i}v_i''\right]f_i'\\ &+\frac{W_i^2}{2}(\boldsymbol{\alpha}_\mathbf{i}^T - \mathbf{d}_\mathbf{i}^T)\boldsymbol{\Omega}(\boldsymbol{\alpha}_\mathbf{i} - \mathbf{d}_\mathbf{i})v_i'^2\ f_i''\end{aligned} \qquad \text{(A5)}$$

The expected utility is the sum of two terms:

---

[1] Our derivations could be made a bit more formal by taking **r**=ξ**r'** and letting ξ become very small. This is a direct application of the "compact" derivations of the approximation by Samuelson (1970).

$EU_i =$ (A3) + (A5)

Then:

$$EU_i \approx v_i(0) + W_i \boldsymbol{\alpha}_\mathbf{i}^\mathbf{T} \bar{\mathbf{r}} v_i' + \frac{W_i^2}{2}\left[\boldsymbol{\alpha}_\mathbf{i}^T \boldsymbol{\Omega} \boldsymbol{\alpha}_\mathbf{i}\right] v_i''$$
$$+\left[W_i(\boldsymbol{\alpha}_\mathbf{i}^T - \mathbf{d}_\mathbf{i}^T)\bar{\mathbf{r}} v_i' + \frac{W_i^2}{2}\boldsymbol{\alpha}_\mathbf{i}^T \boldsymbol{\Omega}\boldsymbol{\alpha}_\mathbf{i} v_i'' - \frac{W_i^2}{2}\mathbf{d}_\mathbf{i}^T \boldsymbol{\Omega}\mathbf{d}_\mathbf{i} v_i''\right] f_i' \quad \text{(A6)}$$
$$+\frac{W_i^2}{2}(\boldsymbol{\alpha}_\mathbf{i}^T - \mathbf{d}_\mathbf{i}^T)\boldsymbol{\Omega}(\boldsymbol{\alpha}_\mathbf{i} - \mathbf{d}_\mathbf{i}) v_i'^2 f_i''$$

Let's compute the optimal allocation by setting to zero the derivative of $EU_i(.)$ with respect to $\boldsymbol{\alpha}_\mathbf{i}$. The first order condition without constraints on $\boldsymbol{\alpha}_\mathbf{i}$:

$$W_i[\bar{\mathbf{r}} v_i' + W_i \boldsymbol{\Omega}\boldsymbol{\alpha}_\mathbf{i} v_i'' + \bar{\mathbf{r}} v_i' f_i' + W_i \boldsymbol{\Omega}\boldsymbol{\alpha}_\mathbf{i} v_i'' f_i' + W_i \boldsymbol{\Omega}(\boldsymbol{\alpha}_\mathbf{i} - \mathbf{d}_\mathbf{i}) v_i'^2 f_i''] = 0$$

The optimal allocation by home investors is:

$$\boldsymbol{\alpha}_\mathbf{i}^* = -\frac{\boldsymbol{\Omega}^{-1}\bar{\mathbf{r}} v_i' \times (1+f_i')}{W_i(v_i'' + v_i'' f_i' + v_i'^2 f_i'')} + \frac{\mathbf{d}_\mathbf{i} v_i'^2 f_i''}{v_i'' + v_i'' f_i' + v_i'^2 f_i''}$$
$$= \boldsymbol{\Omega}^{-1}\frac{\bar{\mathbf{r}}}{\lambda_i} \times \left(1 - \frac{\gamma_i/\lambda_i}{1+\gamma_i/\lambda_i}\right) + \frac{\gamma_i/\lambda_i}{1+\gamma_i/\lambda_i}\mathbf{d}_\mathbf{i} \quad \text{(A7)}$$
$$= \boldsymbol{\Omega}^{-1}\frac{\bar{\mathbf{r}}}{\lambda_i} \times (1-\theta_i) + \theta_i \mathbf{d}_\mathbf{i}$$

where $\lambda_i = -\frac{W_i v_i''}{v_i'}$ is the traditional measure of relative risk aversion and the parameter θ_i can be regarded as "normalized" home preference. Following Bell (1983), we define $\gamma_i = -\frac{W_i v_i' f_i''}{1+f_i'}$ as the foreign aversion parameter and $\theta_i = \frac{\gamma_i/\lambda_i}{1+\gamma_i/\lambda_i}$ as the measure of home preference.

C. Global market equilibrium: Asset pricing relation

The optimal allocations are:

$$\boldsymbol{\alpha}_\mathbf{i}^* = \boldsymbol{\Omega}^{-1}\frac{\bar{\mathbf{r}}}{\lambda_i} \times (1-\theta_i) + \theta_i \mathbf{d}_\mathbf{i}$$

Thus:

$$\bar{\mathbf{r}} = \lambda_i \boldsymbol{\Omega}(\boldsymbol{\alpha}_\mathbf{i}^* - \theta_i \mathbf{d}_\mathbf{i}) / (1-\theta_i)$$

The vector $\mathbf{M}$ is the column vector of $m_i = M_i / M = M_i / W$.

$$\boldsymbol{\alpha}_\mathbf{i}^* = \boldsymbol{\Omega}^{-1}\frac{\bar{\mathbf{r}}}{\lambda_i} \times (1-\theta_i) + \theta_i \mathbf{d}_\mathbf{i}$$
$$\sum_{i=1}^{n} W_i \boldsymbol{\alpha}_\mathbf{i}^* / W = \sum_{i=1}^{n} w_i \times (1-\theta_i)\left[\boldsymbol{\Omega}^{-1}\frac{\bar{\mathbf{r}}}{\lambda_i}\right] + \sum_{i=1}^{n} w_i \theta_i \mathbf{d}_\mathbf{i}$$

We define the world-average home preference: $\theta_W = \sum_{i=1}^{N} w_i \theta_i$.

Let's make the simplifying assumption that [2] $\lambda_i = \lambda$:

$$\mathbf{M} = \left[\mathbf{\Omega}^{-1} \frac{\overline{\mathbf{r}}}{\lambda} \times (1-\theta_W) \right] + \sum_{i=1}^{n} w_i \theta_i \mathbf{d_i}$$

$$\overline{\mathbf{r}} = \lambda \mathbf{\Omega} (\mathbf{M} - \sum_{i=1}^{n} w_i \theta_i \mathbf{d_i}) / (1-\theta_W)$$

Let's define:

$$\delta_i (1-\theta_W) = w_i \theta_i - m_i \theta_W$$

$$\sum_{i=1}^{n} w_i \theta_i \mathbf{d_i} = \sum_{i=1}^{n} m_i \theta_W \mathbf{d_i} + \sum_{i=1}^{n} \delta_i (1-\theta_W) \mathbf{d_i}$$

Remember that the i^{th} element of **M** is m_i, so:

$$\sum_{i=1}^{n} w_i \theta_i \mathbf{d_i} = \theta_W \mathbf{M} + \sum_{i=1}^{n} \delta_i (1-\theta_W) \mathbf{d_i}$$

$$\overline{\mathbf{r}} = \lambda \mathbf{\Omega} (\mathbf{M} - \theta_W \mathbf{M} - \sum_{i=1}^{n} \delta_i (1-\theta_W) \mathbf{d_i}) / (1-\theta_W)$$

$$\overline{\mathbf{r}} = \lambda \mathbf{\Omega} (\mathbf{M} - \sum_{i=1}^{n} \delta_i \mathbf{d_i}) = \lambda \mathbf{\Omega} (\mathbf{M} - \mathbf{\Delta}) \qquad \text{(A8)}$$

Note that $\mathbf{\Delta} = \sum_{i=1}^{n} \delta_i \mathbf{d_i}$ can be considered as a pure arbitrage portfolio as the weights sum to zero while the weights of the market portfolio sum to one.

Let's define:

$$R_W = \sum_{i=1}^{N} m_i R_i \text{ and } R_\delta = \sum_{i=1}^{N} \delta_i R_i$$

For country *i*:

$$E(R_i) - R_0 = \lambda \operatorname{cov}(R_i, R_W) - \lambda \operatorname{cov}(R_i, R_\delta)$$

$$E(R_i) - R_0 = \lambda \operatorname{cov}(R_i, R_W) - \lambda \sum_j \delta_j \operatorname{cov}(R_i, R_j)$$

---

[2] We could possibly use different λ_i and define $\sum \left[\frac{w_i}{\lambda_i} \right] \times (1-\theta_i) = \frac{1}{\lambda} (1-\theta_W)$