Detailed Appendix for: "On the term structure of default premia in the swap and LIBOR markets"

Pierre Collin-Dufresne* Bruno Solnik†

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1 Derivation of the discount factor $P^{\tau}(t)$

$$P^{\tau}(t) = \mathrm{E}_t^{\mathcal{Q}}\left[e^{\int_t^T -r(s)ds}\right] \tag{1.1}$$

$P^{\tau}(t)$ is easily calculated using the fact that $\{r\}$ is a gaussian process. The following can be found, for example, in Duffie and Kan (1996) or Jegadeesh and Pennachi (1996) and is recalled here for completeness.

$$r(s) = r(t)\gamma_r(t,s) + \int_t^s \gamma_r(u,s)\kappa_r(u)\theta(u)du + \int_t^s \gamma_r(u,s)\sigma_r(u)dw_r(u) \tag{1.2}$$

$$\theta(u) = \theta(t)\gamma_\theta(t,u) + \int_t^u \gamma_\theta(v,u)\kappa_\theta(v)\overline{\theta}(v)dv + \int_t^u \gamma_\theta(v,u)\sigma_\theta(v)dw_\theta(v) \tag{1.3}$$

where we define:

$$\gamma_x(t,s) = e^{\int_t^s -\kappa_x(u)du} \quad x = \{r,\theta\}$$

Since θ and r are gaussian processes, $\int_t^T r(s)ds$ is normally distributed with mean and variance:

$$\begin{aligned}
\mathrm{E}_t^{\mathcal{Q}}\left[\int_t^T r(s)ds|r(t)\right] &= r(t)B_r(t,T) + \theta(t)C(t,T) + \int_t^T C(v,T)\kappa_\theta(v)\overline{\theta}(v)dv \\
\mathrm{Var}_t^{\mathcal{Q}}\left[\int_t^T r(s)ds|r(t)\right] &= \int_t^T B_r(u,T)^2\sigma_r^2(u)du + \int_t^T C(v,T)^2\sigma_\theta^2(v)dv + \\
&\quad 2\int_t^T C(u,T)B_r(u,T)\sigma_r(u)\sigma_\theta(u)\rho_{r\theta}du
\end{aligned}$$

where:

$$B_r(t,T) = \int_t^T \gamma_r(t,s)ds$$

*Assistant Professor at Carnegie Mellon University, GSIA 315A 5000 Forbes Avenue, Pittsburgh PA 15213, tel:(412)-2682606, email:dufresne@andrew.cmu.edu

†Professor at HEC, School of Management. Groupe HEC, 78351 Jouy-en-Josas cedex, France. tel: (33)139677284, Fax: (33)139677085, email: solnik@hec.fr

$$
\begin{aligned}
C(t,T) &= \int_t^T c(t,s)ds \\
c(t,s) &= \int_t^s \gamma_r(u,s)\gamma_\theta(t,u)\kappa_r(u)du
\end{aligned}
$$

The discount-factor value is then easily derived from the Laplace transform for normal random variables.

$$
\begin{aligned}
P^\tau(t) &= e^{A_r(t,T)-B_r(t,T)r(t)-C(t,T)\theta(t)} & (1.4)\\
A_r(t,T) &= -\int_t^T C(v,T)\kappa_\theta(v)\overline{\theta}(v)dv + \frac{1}{2}\int_t^T B_r(u,T)^2\sigma_r^2(u)du + \frac{1}{2}\int_t^T C(v,T)^2\sigma_\theta^2 dv + \\
&\quad \int_t^T C(u,T)B_r(u,T)\sigma_r(u)\sigma_\theta(u)\rho_{r\theta}du & (1.5)
\end{aligned}
$$

2 The Risky Zero Coupon Bond price

For the purpose of our implementation we suppose default is triggered by a point process with stochastic intensity $\iota(s)$.[1] Thus, as in Duffie, Shroder and Skiadas (1996), default occurs at an unpredictable stopping time T_D, which in terms of the related counting process $D(t)$ can be defined as $T_D = \{\inf t \ / \ D(t) = 1\}$. We assume that when the firm defaults, bond holders loose a fraction $l(T_D^-)$ of the market value of the security (just prior to default). The value of the corporate zero coupon bond is given by:

$$
e^{-\int_0^t r(u)du} P_L^\tau(t) = \mathrm{E}_{\mathrm{t}}^{\mathcal{Q}}\left[\int_t^{T\wedge T_D} e^{-\int_0^s r(u)du}(1-l(s^-))P_L^{T-s}(s^-)dD(s) + e^{-\int_0^T r(u)du}1_{\{T_D>T\}}\right] \tag{2.1}
$$

where $1_{\{A\}}$ is the indicator function that takes on the value of one if event A is realized and 0 else. Duffie and Singleton (1995) and Duffie, Schroder and Skiadas (1996) have shown that, under certain technical conditions, the solution to that recursive stochastic differential equation could be written simply as:

$$
P_L^\tau(t) = \mathrm{E}_{\mathrm{t}}^{\mathcal{Q}}\left[e^{-\int_t^T (r(s)+l(s)\iota(s))ds}\right] \quad \forall t < T_D \tag{2.2}
$$

Thus the credit spread defined in the text is simply $\delta = l\iota$. In other words, δ is the instantaneous expected loss rate, the product of the default intensity and the proportion of value lost in bankruptcy. This product may itself be stochastic. It may depend on different state variables and may incorporate jumps.[2]

In fact, we argue that it is necessary to put more structure on the dynamics of the instantaneous credit spread when modeling the term structure of credit spreads. We assume that for a top-rated credit issuer at time t its instantaneous credit process has the following dynamics:

$$
\begin{aligned}
d\delta^t(s) &= \kappa_\delta(s)\left(\overline{\delta}^t(s) - \delta^t(s)\right)ds + \sigma_\delta(s)dw_\delta(s) + \nu_1^u(s)dN_u^t(s) - \nu_1^d(s)dN_d^t(s) & (2.3)\\
d\overline{\delta}^t(s) &= \nu_2^u(s)dN_u^t(s) - \nu_2^d(s)dN_d^t(s) & (2.4)\\
\delta^t(t) &= \delta(t) & (2.5)\\
\overline{\delta}^t(t) &= \overline{\delta} & (2.6)
\end{aligned}
$$

[1]See Brémaud (1981) for the definition of such a process (D7 p. 27).

[2]However, it is important that the no-default value of the contract does not jump at the time of bankruptcy. In other words, the jumps in δ should occur with zero probability at the default event. Duffie, Schroder and Skiadas (1996) provide solutions to defaultable bond prices for more complicated cases.

where N_d^t, N_u^t are independent counting processes with deterministic intensity λ_u, λ_d and $N_d^t(t) = N_u^t(t) = 0$. $\kappa_\delta, \sigma_\delta, \nu_u^1, \nu_u^2, \nu_d^1, \nu_d^2$ are at most deterministic functions of time and $\overline{\delta}$ is constant, so that the instantaneous credit spread for a "refreshed credit quality issuer" follows:

$$d\delta(s) = \kappa_\delta(s)\left(\overline{\delta} - \delta(s)\right)ds + \sigma_\delta(s)dw_\delta(s) \tag{2.7}$$

Thus the price of the risky zero-coupon bond price for a top-rated credit issuer at time t is:

$$P_L^\tau(t) = \mathrm{E}_\mathrm{t}^\mathcal{Q}\left[e^{-\int_t^T(r(s)+\delta^t(s))ds}\right] \tag{2.8}$$

The expectation can be solved via standard PDE approach as in Duffie and Kan (1996), or Das and Foresi (1996). We provide here another equivalent proof using standard martingale techniques.

First notice that we can write:

$$\delta^t(s) = \delta(s) + \int_t^s(\nu_2^u(v) + (\nu_1^u(v) - \nu_2^u(v))\gamma_{\kappa_\delta}(v,s))dN_u^t(v) - \int_t^s(\nu_2^d(v) + (\nu_1^d(v) - \nu_2^d(v))\gamma_{\kappa_\delta}(v,s))dN_d^t(v) \tag{2.9}$$

where δ is the Ornstein-Uhlenbeck process of equation (2.7). Assuming that (N_d^t, N_u^t) is independent of $(w_r, w_\delta, w_\theta)$ we obtain:

$$P_L^\tau(t) = \mathrm{E}_\mathrm{t}^\mathcal{Q}\left[e^{-\int_t^T(r(s)+\delta(s))ds}\right]\mathrm{E}_\mathrm{t}^\mathcal{Q}\left[e^{-\int_t^T\int_t^s(\nu_2^u(v)+(\nu_1^u(v)-\nu_2^u(v))\gamma_{\kappa_\delta}(v,s))dN_u^t(v)ds}\right]\mathrm{E}_\mathrm{t}^\mathcal{Q}\left[e^{+\int_t^T\int_t^s(\nu_2^d(v)+(\nu_1^d(v)-\nu_2^d(v))\gamma_{\kappa_\delta}(v,s))dN_d^t(v)ds}\right] \tag{2.10}$$

To value the first expectation we apply a forward neutral change of measure introduced by El Karoui and Rochet (1989) and Jamshidian (1991) and defined by $\xi = \frac{d\mathcal{Q}^T}{d\mathcal{Q}} = \frac{e^{-\int_0^T r_s ds}}{P^T(0)}$. Note that under the $\mathcal{Q}^T$-measure, δ has the following dynamics:

$$d\delta(t) = \left(\kappa_\delta(t)(\overline{\delta} - \delta(t)) - \rho_{r\delta}\sigma_\delta(t)\sigma_r(t)B_r(t,T) - \rho_{\theta\delta}\sigma_\delta(t)\sigma_\theta(t)C(t,T)\right)dt + \sigma_\delta(t)dw_\delta^{\mathcal{Q}^T}(t) \tag{2.11}$$

This follows from Girsanov's theorem which states that given the Radon-Nikodym derivative ξ above, under $\mathcal{Q}^T$, one can define three standard brownian motions $w_r^{\mathcal{Q}^T}, w_\theta^{\mathcal{Q}^T}$ and $w_\delta^{\mathcal{Q}^T}$ with the same correlation structure $\rho_{r\theta}, \rho_{r\delta}, \rho_{r\delta}$ as follows: $dw_r^{\mathcal{Q}^T} = dw_r(t) + (\sigma_r(t)B_r(t,T) + \rho_{r\theta}\sigma_\theta C(t,T))\,dt$, $dw_\delta^{\mathcal{Q}^T} = dw_\delta(t) + (\rho_{r\delta}\sigma_r(t)B_r(t,T) + \rho_{r\theta}\sigma_\theta(t)C(t,T))\,dt$, and $dw_\theta^{\mathcal{Q}^T} = dw_\theta(t) + (\rho_{r\theta}\sigma_r(t)B_r(t,T) + \sigma_\theta(t)C(t,T))\,dt$.[3]

We see that $\{\delta\}$ is a gaussian process under $\mathcal{Q}^T$. Using the same approach as for the derivation of riskless bonds (see appendix 1) we obtain:

$$\mathrm{E}_\mathrm{t}^\mathcal{Q}\left[e^{-\int_t^T(r(s)+\delta(s))ds}\right] = \mathrm{E}_\mathrm{t}^\mathcal{Q}\left[e^{-\int_t^T r(s)ds}\right]\mathrm{E}_\mathrm{t}^{\mathcal{Q}^\mathrm{T}}\left[e^{-\int_t^T \delta(s)ds}\right] = P^\tau(t)e^{A_\delta(t,T)-B_\delta(t,T)\delta(t)} \tag{2.12}$$

$$\begin{aligned}
A_\delta(t,T) &= -\int_t^T B_\delta(u,T)\left(\kappa_\delta(u)\overline{\delta} - \rho_{r\delta}\sigma_\delta(u)\sigma_r(u)B_r(u,T) - \rho_{r\theta}\sigma_\delta(u)\sigma_\theta(u)C(u,T)\right)du + \\
&\quad 1/2\int_t^T B_\delta(u,T)^2\sigma_\delta^2(u)du \\
B_\delta(t,T) &= \int_t^T \gamma_\delta(t,s)ds \\
\gamma_\delta(t,s) &= e^{\int_t^s -\kappa_\delta(u)du}
\end{aligned}$$

[3] The change in drift comes from the process of the Girsanov factor process defined by: $\xi_t = \mathrm{E}_\mathrm{t}^\mathcal{Q}[\xi]$. Notice that $\frac{d\xi_t}{\xi_t} = -\sigma_r(t)B_r(t,T)dw_r(t) - \sigma_\theta(t)C(t,T)dw_\theta(t)$ and the change in the drift for δ corresponds precisely to the diffusion part of ξ_t times the correlation. See Karatzas and Shreve (1991) for an exposition of Girsanov's theorem. Girsanov's theorem is, in general, stated for independent Brownian motions, but can be straightforwardly extended to the case of correlated brownian motions.

To value the expectation

$$\mathrm{E}_{\mathrm{t}}^{\mathcal{Q}}\left[e^{-\int_t^T\int_t^s(\nu_2^u(v)+(\nu_1^u(v)-\nu_2^u(v))\gamma_{\kappa_\delta}(v,s))dN_u^t(v)ds}\right]$$

we use Fubini's theorem to show that

$$\int_t^T\int_t^s(\nu_2^u(v)+(\nu_1^u(v)-\nu_2^u(v))\gamma_{\kappa_\delta}(v,s))dN_u^t(v)ds=\int_t^T(\nu_2^u(v)(T-v)+(\nu_1^u(v)-\nu_2^u(v))B_\delta(v,T))\,dN_u^t(v)$$

We also use the following lemma:

Lemma 1 *We claim:*

$$\mathrm{E}_{\mathrm{t}}^{\mathcal{Q}}\left[e^{-\int_t^T f(s)dN(s)}\right]=\mathrm{E}_{\mathrm{t}}^{\mathcal{Q}}\left[e^{-\int_t^T(1-e^{-f(s)})\lambda(s)ds}\right]\tag{2.13}$$

where $\{N\}$ *is a point process with* $\mathcal{Q}$*-intensity* $\lambda(\cdot)$ *and* $f(\cdot)$ *is a predictable and bounded process on* $[t,T]$.

Proof: See E17 page 73 in Brémaud (1981) - but notice there is a typo in (4.9), (4.11) page 73 - the correct statement is as above, for which we provide a sketch of proof below.

We will show that $M(t)=\exp\left(-\int_0^t f(s)dN(s)+\int_0^t(1-e^{-f(s)})\lambda(s)ds\right)$ is a $\mathcal{Q}$-martingale. Applying generalized Itô's lemma to $M(t)$ (see Protter 1995) we obtain:

$$dM(t)=\lambda(t)(1-e^{-f(t)})M(t^-)dt+\Delta M(t)\tag{2.14}$$

But $\Delta M(t)=M(t)-M(t^-)=M(t^-)(e^{-f(t)}-1)dN(t)$. Hence,

$$dM(t)=-(1-e^{-f(t)})M(t^-)(dN(t)-\lambda(t)dt)\tag{2.15}$$

But by the definition of intensity $N(t)-\int_0^t\lambda(s)ds$ is a $\mathcal{Q}$ martingale. Since $f(\cdot)$ is predictable and bounded, this implies that $M(t)$ is also a $\mathcal{Q}$ Martingale. It follows that $M(t)=\mathrm{E}_{\mathrm{t}}^{\mathcal{Q}}[M(T)]$ and we obtain:

$$\mathrm{E}_{\mathrm{t}}^{\mathcal{Q}}\left[e^{-\int_t^T f(s)dN(s)}\right]=\mathrm{E}_{\mathrm{t}}^{\mathcal{Q}}\left[e^{-\int_t^T(1-e^{-f(s)})\lambda(s)ds}\right]\tag{2.16}$$

□

Using the above, we obtain:

$$\mathrm{E}_{\mathrm{t}}^{\mathcal{Q}}\left[e^{-\int_t^T\int_t^s(\nu_2^u(v)+(\nu_1^u(v)-\nu_2^u(v))\gamma_{\kappa_\delta}(v,t))dN_u^t(v)ds}\right]=\mathrm{E}_{\mathrm{t}}^{\mathcal{Q}}\left[e^{-\int_t^T(1-e^{-f_u(v,T)})\lambda_u^t(v)dv}\right]\tag{2.17}$$

where

$$f_u(v,T)=\nu_2^u(v)(T-v)+(\nu_1^u(v)-\nu_2^u(v))B_\delta(v,T)$$

Similarly we obtain:

$$\mathrm{E}_{\mathrm{t}}^{\mathcal{Q}}\left[e^{\int_t^T\int_t^s(\nu_2^d(v)+(\nu_1^d(v)-\nu_2^d(v))\gamma_{\kappa_\delta}(v,t))dN_d^t(v)ds}\right]=\mathrm{E}_{\mathrm{t}}^{\mathcal{Q}}\left[e^{-\int_t^T(1-e^{-f_d(v,T)})\lambda_d^t(v)dv}\right]\tag{2.18}$$

where

$$f_d(v,T)=-\nu_2^d(v)(T-v)-(\nu_1^d(v)-\nu_2^d(v))B_\delta(v,T)$$

Putting equations (2.10),(2.12), (2.17), and (2.18) together, we finally obtain the formula for risky zero coupon LIBOR bond prices:

$$P_L^\tau(t) = P^\tau(t) P_\delta^\tau(t) e^{\int_t^T -\mu^t(s,T)ds} \tag{2.19}$$

where:

$$P^\tau(t) = e^{A_r(t,T) - B_r(t,T)r(t) - C(t,T)\theta(t)} \tag{2.20}$$

$$P_\delta^\tau(t) = e^{A_\delta(t,T) - B_\delta(t,T)\delta(t)} \tag{2.21}$$

$$\mu^t(s,T) = \lambda_u^t(s)\left(1 - e^{-f_u(s,T)}\right) + \lambda_d^t(s)\left(1 - e^{-f_d(s,T)}\right)$$

This model has interesting implications for the top-rated credit quality credit spread. For simplicity assume $\nu_1^u = \nu_2^u = \nu_u$ and a $\nu_1^d = \nu_2^d = \nu_d$ and that all coefficients are constant. In that case

$$\mu^t(s,T) = \lambda_u\left(1 - e^{-\nu_u(T-s)}\right) + \lambda_d\left(1 - e^{\nu_d(T-s)}\right)$$

And we see that if both probability and size of credit deterioration are greater than that of credit risk appreciation ($\lambda_u \geq \lambda_d$ and $\nu_u \geq \nu_d$) then the term structure of credit spread will be increasing in time to maturity for top-rated issuers. Notice that to a first order

$$\mu^t(s,T) = (\lambda_u\nu_u - \lambda_d\nu_d) * (T-s)$$

i.e. only the expected appreciation net of expected depreciation 'matters.' For our empirical application we focus on estimating this difference.

Defining the term structure of credit spreads, $SP^\tau(t)$, to be the difference between the yield on a defaultable bond and a risk-free bond with similar maturity, we obtain for the present model:

$$SP^\tau(t) \equiv \frac{-1}{\tau}\left(\ln P_L^\tau(t) - \ln P^\tau(t)\right) = \frac{-A_\delta(t,T) + B_\delta(t,T)\delta(t) + \int_t^T \mu^t(s,T)ds}{\tau} \tag{2.22}$$

Notice that for the case of constant coefficients we obtain the two following limiting results:

$$\lim_{\tau\to 0} SP^\tau(t) = \delta(t) \ \text{ and } \ \lim_{\tau\to+\infty} SP^\tau(t) = \overline{\delta} - \frac{\sigma_\delta^2}{2\kappa_\delta^2} + \lambda_u - \lambda_d \tag{2.23}$$

This illustrates that as the maturity tends to zero the spread tends to the instantaneous credit spread for "refreshed credit quality" top-rated issuers. On the other hand, as maturity increases towards infinity, the spread tends towards the limiting spread of a "refreshed credit quality" issuer plus the instantaneous probability of credit depreciation minus the instantaneous probability of credit appreciation.[4]

3 Derivation of the swap rate formula

The swap rate $Y_S^\tau(t)$ solves:

$$\mathrm{E}_{\mathrm{t}}^{\mathcal{Q}}\left[\sum_{i=1}^{i=2n} e^{-\int_t^{t_i} r(s)ds} Y_S^\tau(t)\right] = \mathrm{E}_{\mathrm{t}}^{\mathcal{Q}}\left[\sum_{i=1}^{i=2n} e^{-\int_t^{t_i} r(s)ds} Y_L^{.5}(t_{i-1})\right] \tag{3.1}$$

[4] The intuition for the fact that the size of the jump in credit spread ν does not appear in the formula, is that only the probability of no jumps occurring which depends on the survival probability $e^{-\int_0^T \lambda_s ds}$ affects spreads as $T \to \infty$. Indeed, conditional on jumps occurring, the instantaneous expected risk-adjusted rate grows linearly in time, which drives the bond price (conditional on jumps occurring) to zero at a rate e^{-T^2}.

Using the fact that the swap rate is a constant, the definition of the risk-free discount bond (1.4) and the definition of the (6-month) LIBOR rate, we obtain:

$$Y_S^\tau(t) \sum_{i=1}^{i=2n} P^{.5i}(t) = \sum_{i=1}^{i=2n} \mathrm{E}_{\mathrm{t}}^{\mathcal{Q}} \left[e^{-\int_t^{t_i} r(s)ds} \frac{1 - P_L^{.5}(t_{i-1})}{P_L^{.5}(t_{i-1})} \right] \tag{3.2}$$

Substituting the expression for the risky discount bond and simple algebra lead to:

$$(1 + Y_S^\tau(t)) \sum_{i=1}^{i=2n} P^{.5i}(t) = \sum_{i=1}^{i=2n} \mathrm{E}_{\mathrm{t}}^{\mathcal{Q}} \left[\frac{e^{-\int_t^{t_i} r(s)ds}}{P^{0.5}(t_{i-1}) P_\delta^{0.5}(t_{i-1}) e^{\int_{t_{i-1}}^{t_i} -\mu^{t_{i-1}}(s,t_i)ds}} \right] \tag{3.3}$$

Conditioning with respect to t_{i-1} inside the expectation, and using the fact that $\mu(\cdot)$ is deterministic, we obtain:

$$(1 + Y_S^\tau(t)) \sum_{i=1}^{i=2n} P^{.5i}(t) = \sum_{i=1}^{i=2n} \mathrm{E}_{\mathrm{t}}^{\mathcal{Q}} \left[\frac{e^{-\int_t^{t_{i-1}} r(s)ds}}{P_\delta^{0.5}(t_{i-1})} \right] e^{\int_{t_{i-1}}^{t_i} \mu^{t_{i-1}}(s,t_i)ds} \tag{3.4}$$

We thus need to compute the following expectation: $\mathrm{E}_{\mathrm{t}}^{\mathcal{Q}} \left[\frac{e^{-\int_t^{t_{i-1}} r(s)ds}}{P_\delta^{.5}(t_{i-1})} \right]$

We use the forward-neutral change of measure described above and the formula of P_δ to obtain:

$$\begin{aligned} \mathrm{E}_{\mathrm{t}}^{\mathcal{Q}} \left[\frac{e^{-\int_t^{t_{i-1}} r(s)ds}}{P_\delta^{.5}(t_{i-1})} \right] &= \mathrm{E}_{\mathrm{t}}^{\mathcal{Q}} \left[e^{-\int_t^{t_{i-1}} r(s)ds} \right] \mathrm{E}_{\mathrm{t}}^{\mathcal{Q}^{\mathrm{T}}} \left[e^{-A_\delta(t_{i-1},t_i) + B_\delta(t_{i-1},t_i)\delta_{t_{i-1}}} \right] \\ &= P^{t_{i-1}}(t) \mathrm{E}_{\mathrm{t}}^{\mathcal{Q}^{\mathrm{T}}} \left[e^{-A_\delta(t_{i-1},t_i) + B_\delta(t_{i-1},t_i)\delta_{t_{i-1}}} \right] \end{aligned}$$

Using the fact that δ is a gaussian process under the forward-neutral measure, this last expression is easily computed. Noting that $B_\delta(t,t_i) - B_\delta(t,t_{i-1}) = \gamma_\delta(t,t_{i-1}) B_\delta(t_{i-1},t_i)$ and tedious algebra then lead to equation the following:

$$1 + Y_S^\tau(t) = \frac{\sum_{i=1}^{i=2n} \left[P^{.5(i-1)}(t) P_\delta^{.5(i-1)}(t) / P_\delta^{.5i}(t) * e^{\int_{t_{i-1}}^{t_i} \mu^{t_{i-1}}(s,t_i)ds} * \mathcal{C}(t,t_{i-1},t_i) \right]}{\sum_{i=1}^{i=2n} P^{.5i}(t)} \tag{3.5}$$

Where in the above, we have defined:

$$\begin{aligned} \ln \mathcal{C}(t,t_{i-1},t_i) &= \int_t^{t_{i-1}} B_\delta(u,t_{i-1}) \left(B_r(u,t_i) - B_r(u,t_{i-1}) \right) \rho_{r\delta} \sigma_r(u) \sigma_\delta(u) du + \\ &\quad \int_t^{t_{i-1}} B_\delta(u,t_{i-1}) \left(C(u,t_i) - C(u,t_{i-1}) \right) \rho_{\theta\delta} \sigma_\theta(u) \sigma_\delta(u) du + \\ &\quad \int_t^{t_{i-1}} B_\delta(u,t_i) \left(B_\delta(u,t_i) - B_\delta(u,t_{i-1}) \right) \sigma_\delta(u)^2 du \end{aligned} \tag{3.6}$$

After some rearranging and simple algebra, we obtain the equation for the swap rate in the text with:

$$\ln \mathcal{C}' = \int_{t_{i-1}}^{t_i} \mu^{t_{i-1}}(s,t_i)ds - \left(\int_t^{t_i} \mu^t(s,t_i)ds - \int_t^{t_{i-1}} \mu^t(s,t_{i-1})ds \right) \tag{3.7}$$

4 Proof of Proposition 1

To prove the proposition notice that if $\mathcal{C} = 1$ the only difference between the LIBOR bond yield rate formula and the swap rate is the weightings of the forward LIBOR rates. It follows from the definitions of ω_i^L and ω_i that $\omega_i < \omega_i^L$ if and only if $P_\delta^{.5i}(t) > \frac{\sum_{j=1}^{j=2n} P_\delta^{.5j}(t) P^{.5j}(t)}{\sum_{j=1}^{j=2n} P^{.5j}(t)}$. But $P_\delta^\tau(t)$ is a strictly decreasing function of time to maturity. Hence there exists a $\bar{i}$ verifying $0 < \bar{i} < n$ such that for all $i < \bar{i}$ the inequality is verified. In other words, the weigthings in the formula to compute par-bond rates put more weight on short-maturity forward LIBOR rates relative to the weights in the swap rate formula and the inverse is true for long-maturity forward LIBOR rates. If the forward LIBOR curve is upward-sloping, then the weighted sum involved in computing par-bonds rates weighs more the low forward LIBOR rates and less the high forward LIBOR rates relative to the sum involved in computing the swap rates. As a consequence, the LIBOR bond yield is lower than the swap rate. The inverse is true if the forward LIBOR curve is downward-sloping. And obviously the swap rates and LIBOR yields are equal if the forward LIBOR curve is flat.

5 The empirical implementation

5.1 Data

We use weekly data for Treasury, LIBOR par-bond and swap rates from October 12, 1988 to January 29 1997. The data were obtained from Datastream. Datastream reports the mid swap rates[5] quoted by a major swap dealer for maturities of 2, 3, 4, 5, 7 and 10 years. Treasury bond data covers the maturities: 1, 2, 3, 4, 5, 7 and 10 years. Finally we use the LIBOR yields reported by Datastream for maturities 0.5, 1, 2, 3, 4, and 5 years. These are quoted yields for fixed-coupon par-bonds negotiated OTC and issued by corporate issuers (usually banks and financial institutions) rated AA or better.[6]

We report in Table 1 average Treasury, swap and LIBOR rates (based on semi-annual compounding) for the period 1988 to 1997 as well as spreads. While the Treasury yield curve has assumed many shapes over the period, on average, it has been upward-sloping. The average spread between LIBOR and Treasury yields is increasing with maturity, it ranges from 38 bp at a 1-year maturity to 51 bp at a 5-year maturity. The average spread between LIBOR and swap rates increases from 14.7 bp for 2 years to 17.3 bp for 5 years. However this spread is quite volatile and can be negative in some periods. By construction, the swap rate for a 6-month swap should be equal to the 6-month LIBOR (the floating-rate leg). Hence the LIBOR-Swap spread is zero for a 6-month maturity.

5.2 Econometric methodology

In order to subject our model to empirical scrutiny, we make a few simplifying assumptions. We assume that all parameters are constant. To reduce the number of parameters to be estimated, we simply assume that $\nu_1^u = \nu_2^u = \nu_u$, $\nu_1^d = \nu_2^d = \nu_d$ and that $\lambda_d^t(s) = \lambda_d$, and $\lambda_u^t(s) = \lambda_u$. In words, we assume that when the credit quality deteriorates (appreciates), both the long-term mean and the level of the credit spreads jump by an equal amount, and that the probability of credit deterioration (appreciation) is constant.

[5]The bid and ask swap rates quoted depend on the credit quality of the customer. The bid-mid and mid-ask spreads for a generic swap quoted to a AAA or AA customer are generally equal to one basis point over the period. As mentioned in Sun, Sundaresan and Wang (1993) and Cossin and Pirotte (1997), the spreads increase by a few basis points for a lesser-rated customer.

[6]The market is pretty liquid, see Sun, Sundaresan and Wang (1993) for a discussion of the LIBOR bond market and comparisons of the Datastream-data with alternative data sets

Table 1: Summary statistics on Treasury yields and various spreads. All yields are semi-annual and reported in per cent.

Maturity	Treasury	LIBOR	Swap	LIBOR-Treasury	Swap-Treasury	LIBOR-Swap
1	5.875	6.256	n.a.	0.381	n.a.	n.a.
2	6.287	6.685	6.538	0.398	0.251	0.147
3	6.504	6.983	6.829	0.479	0.325	0.154
4	6.700	7.211	7.051	0.511	0.350	0.161
5	6.881	7.395	7.222	0.514	0.341	0.173
7	7.075	n.a.	7.453	n.a.	0.377	n.a.
10	7.257	n.a.	7.659	n.a.	0.402	n.a.

Because of a well-known indeterminacy arising in such models (Duffee (1999), Duffie and Singleton (1999)) we cannot estimate the intensity of the jump separately from the size of the jump. Moreover we cannot estimate appreciation seperately from depreciation. We thus estimate the expected appreciation net of expected depreciation $\mu \equiv \lambda_u \nu_u - \lambda_d \nu_d$.[7]

We conduct a maximum-likelihood estimation using both time-series and cross-sectional data in the spirit of Chen and Scott (1993) and Pearson and Sun (1994). The approach consists in using three arbitrarily chosen yields, e.g. a swap rate and a LIBOR bond yield to determine the state (r, θ, δ) using the formulas for the swap rate, the Treasury rate and the corporate bond rate, and given a vector or parameter values. The remaining yields, which, at any point in time, are also deterministic functions of the state variables are then over identified. Following Chen and Scott (1993), we assume these other yields are priced or measured with 'error.'[8]

More formally, we suppose $\hat{Y}(t) = Y(t) + u(t)$, where $Y(t)$ is the vector of "true arbitrage-free" yields and $u(t)$ is a vector of "measurement/pricing errors" that is assumed to be identically zero for three components, say the first three. The remaining components of the measurement-error vector are assumed to be driven by an AR1 process similar to that of Chen and Scott (1993) of the form $u_i(t_j) = \rho_i u_i(t_{j-1}) + \epsilon_i(t_j)$, where the ϵ_i are joint-normally distributed and independent of r, θ and δ. The yields are observed at times $t_0 < t_1 < t_2 < \ldots < t_m$. Because r, θ, δ are jointly gaussian, the likelihood can easily be derived.

Indeed, the likelihood given the Markovian properties of our model, can then be expressed as follows:

$$\mathcal{L} \equiv \mathcal{L}\left(\hat{Y}(t_1), \hat{Y}(t_2), \ldots, \hat{Y}(t_m)\right) = \mathcal{L}\left(\hat{Y}(t_m)|\hat{Y}(t_{m-1})\right) \cdots \mathcal{L}\left(\hat{Y}(t_1)|\hat{Y}(t_0)\right) * \mathcal{L}\left(\hat{Y}(t_0)\right) \quad (5.1)$$

where $\hat{Y}(t_i) = Y(t_i) + u(t_i)$ is a vector of swap rates, LIBOR and Treasury yields observed at time t_i with 'error.' Let us define $\hat{Y}_i(t)$ to be the i^{th} element of $\hat{Y}(t)$. As discussed previously, knowledge of (say) $Y_1(t)$, $Y_2(t)$ and $Y_3(t)$ (i.e. assuming $u_1 = u_2 = u_3 = 0$) allows us to perfectly infer the values of $r(t)$, $\theta(t)$ and $\delta(t)$. Conditioning on the first two elements in $\hat{Y}$, we obtain:

$$\mathcal{L} = \mathcal{L}\left(\hat{Y}(t_0)\right) \Pi_{j=1}^{m} \mathcal{L}\left(\hat{Y}_i(t_j), i > 3|\hat{Y}(t_{j-1}), Y_1(t_j), Y_2(t_j), Y_3(t_j)\right) * \mathcal{L}\left(Y_1(t_j), Y_2(t_j), Y_3(t_j)|\hat{Y}(t_{j-1})\right) \quad (5.2)$$

Since we assume an auto-regressive process for the error terms, we have $\hat{Y}_i(t_j) = Y_i(t_j) + \rho_i u_i(t_{j-1}) +$

[7] With these assumptions $\mu^t(s)$ reduces to: $\mu^t(s,T) = \lambda_u \left(1 - e^{-\nu_u(T-s)}\right) - \lambda_d \left(1 - e^{-\nu_d(T-s)}\right)$. But for small ν (empirically it is of the order of 10^{-4}) a Taylor expansion shows that this reduces to $\mu^t(s,T) = (\lambda_u \nu^u - \lambda_d \nu^d) * (T - s)$. We thus choose to jointly estimate the parameter $\mu \equiv \lambda_u \nu^u - \lambda_d \nu^d$ using the approximation: $\mu^t(s,T) = \mu(T - s)$.

[8] Duffie and Singleton (1997) use a similar method. Alternatively, we could have used a Kalman-filter to avoid making an arbitrary assumption on which yields are priced without errors.

$\epsilon_i(t_j)$ and $Y_i(t)$ $i > 3$ is deterministic given $(Y_1(t), Y_2(t), Y_3(t))$, we obtain:

$$\mathcal{L} = \mathcal{L}\left(\hat{Y}(t_0)\right) \Pi_{j=1}^{m} \mathcal{L}\left(\epsilon(t_j)|\epsilon(t_{j-1})\right) * \mathcal{L}\left(r(t_j), \theta(t_j), \delta(t_j)|r(t_{j-1}), \theta(t_{j-1}), \delta(t_{j-1})\right) * |J(t_j)| \quad (5.3)$$

Where $|J(t)|$ is the jacobian of the transformation from r,θ,δ to Y_1, Y_2, Y_3 at date t (and the Jacobian of the transformation from ϵ to $\hat{Y}$ is one).

Finally, observing that (r, θ, δ) forms a gaussian process, we can easily derive its (gaussian) transition density. With the additional assumption that the (ϵ) are independent normally distributed, the log-likelihood can easily be derived.[9]

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